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## Statistical learning control of uncertain systems: theory and algorithms <sup>☆</sup>

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### Abstract

It has recently become clear that many control problems are too difficult to admit analytic solutions. New results have also emerged to show that the computational complexity of some “solved” control problems is prohibitive. Many of these control problems can be reduced to decidability problems or to optimization questions. Even though such questions may be too difficult to answer analytically, or may not be answered exactly given a reasonable amount of computational resources, researchers have shown that we can “approximately” answer these questions “most of the time”, and have “high confidence” in the correctness of the answers. © 2001 Elsevier Science Inc. All rights reserved.

*Keywords:* Empirical processes; Statistical learning; Robust control; Optimization

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## 1. Introduction

Beginning with the work of Bellman [2], it became clear that many control and optimization problems will not efficiently scale as the dimension of the problem increases. Bellman called this phenomenon the “curse of dimensionality” and ever since, large-size problems became a major concern for both theoreticians and engineers. More recently, it became clearer that the “intrinsic dimension” (such as  $V$ -dimension or  $P$ -dimension) of the problem is what really matters, i.e. a large-dimensional problem may still be efficiently solved while a smaller dimension one may not.

In order to fix ideas in this paper, we establish the following categorizations:

- Which control questions can be answered? or which problems are solvable? This is the realm of Decision Theory, and will give a yes/no answer.
- Which control problems are solvable but difficult? Which problems are solvable but at a prohibitive cost? This is the realm of Computational Complexity Theory, and will tell us which decidable problems are not “practically” solvable.
- What do we do about approximately solving those problems which are costly to solve exactly? This is the realm of Stochastic Algorithms and Statistical Learning Theory [8,9].

Our paper is mainly concerned with the last item above, as a method to deal with those problems where the curse of dimensionality is cast. Many authors have recently advanced the notion of probabilistic methods to solve difficult problems in control analysis [1,6] and design [12]. These methods build on standard Monte Carlo methods (Chernoff Bounds, Hoeffding Inequality, etc.) with ideas advanced during the 1960s and 1970s on the theory of empirical processes and statistical learning. In control theory, some of the original (Monte Carlo) ideas have already been used to solve *robust analysis* problems while more recent research used learning theory to solve *robust control* problems.

Unfortunately, and as acknowledged by the various authors, probabilistic methods, while more efficient than gridding techniques (which suffer from the curse of dimensionality), still require a large number of samples in order to guarantee accurate designs. On the other hand, it was conjectured and verified experimentally that much smaller bounds on the number of samples may be sufficient (tens of thousands instead of millions) to guarantee a certain level of performance. Some researchers in fact used hundreds of samples instead of the millions implied by the theoretical bounds, while acknowledging that the theoretical guarantees of accuracy and confidence no longer hold [11,12]. The question then becomes: what (if any) guarantees are obtained by the smaller number of samples, or more appropriately, is there a small bound on the number of samples which can still guarantee the desired performance?

This paper answers the last question affirmatively, and does so by invoking different versions of *bootstrap sequential learning* algorithms. We show that the necessary number of samples (known as the sample complexity of learning) is a random variable, which is bounded below by the unknown value of sample size at which the algorithm starts to work, and bounded above by bounds of the same order found by earlier research. This will also lead to the notion of *efficient learning times* which is then used to present our results in a computationally attractive manner.

## 2. Overview of randomized algorithms and statistical learning theory

We start with an overview of standard learning theory concepts and results obtained in [11] along with their application to control problems.

Let  $(S, \mathcal{A})$  be a measurable space and let  $\{X_n\}_{n \geq 1}$  be a sequence of independent identically distributed (i.i.d) observations in this space with common distribution  $P$ . We assume that this sequence is defined on a probability space  $(\Omega, \Sigma, \mathbb{P})$ . Denote by  $\mathcal{P}(S) := \mathcal{P}(S, \mathcal{A})$  the set of all probability measures on  $(S, \mathcal{A})$ . Suppose  $\mathcal{P} \subset \mathcal{P}(S)$  is a class of probability distributions such that  $P \in \mathcal{P}$ . In particular, if one has no prior knowledge about  $P$ , then  $\mathcal{P} = \mathcal{P}(S)$ . In this case, we are in the setting of *distribution free learning*. One application of statistical learning theory is to the problem of *risk minimization*. Given a class  $\mathcal{F}$  of  $\mathcal{A}$ -measurable functions  $f$  from  $S$  into  $[0, 1]$  (e.g., decision rules in a pattern recognition problem), the risk functional is defined as

$$R_P(f) := P(f) := \int_S f dP := \mathbb{E}f(X), \quad f \in \mathcal{F}.$$

The goal is to find a function  $f_P$  that minimizes  $R_P$  on  $\mathcal{F}$ . Typically, the distribution  $P$  is unknown and the solution of the risk minimization problem is to be based on a sample  $(X_1, \dots, X_n)$  of certain size  $n$  of independent observations from  $P$ . In this case, the goal of statistical learning is more modest: given  $\epsilon > 0, \delta \in (0, 1)$ , to find an estimate  $\hat{f}_n \in \mathcal{F}$  of  $f_P$ , based on the data  $(X_1, \dots, X_n)$ , such that

$$\sup_{P \in \mathcal{P}} \mathbb{P}\{R_P(\hat{f}_n) \geq \inf_{f \in \mathcal{F}} R_P(f) + \epsilon\} \leq \delta. \tag{1}$$

In other words, one can write that with probability  $1 - \delta, R_P(\hat{f}_n)$  is within  $\epsilon$  of  $\inf_{f \in \mathcal{F}} R_P(f)$ .

A method of *empirical risk minimization* is widely used in learning theory. Namely, the unknown distribution  $P$  is replaced by *the empirical measure*  $P_n$ , defined as

$$P_n(A) := \frac{1}{n} \sum_{k=1}^n I_A(X_k), \quad A \in \mathcal{A},$$

where  $I_A(x) = 1$  for  $x \in A$  and  $I_A(x) = 0$  for  $x \notin A$ . The risk functional  $R_P$  is replaced by the empirical risk  $R_{P_n}$ , defined by

$$R_{P_n}(f) := P_n(f) := \int_S f \, dP_n := \frac{1}{n} \sum_{k=1}^n f(X_k), \quad f \in \mathcal{F}.$$

The problem is now to minimize the empirical risk  $R_{P_n}$  on  $\mathcal{F}$ . Let  $f_{P_n} \in \mathcal{F}$  be a function that minimizes  $R_{P_n}$  on  $\mathcal{F}$ .

In what follows  $f_{P_n}$  is used as our learning algorithm, i.e.  $\hat{f}_n := f_{P_n}$ . Determining the sample complexity of the empirical risk minimization method is definitely one of the central and most challenging problems of statistical learning theory (see, e.g., [4], or [11] for the relevant discussion in the context of robust control problems). A reasonable upper bound for the sample complexity can be obtained by finding the minimal value of  $n$  for which the expected value  $\mathbb{E}f(X)$  is approximated uniformly over the class  $\mathcal{F}$  by the empirical means with given accuracy  $\epsilon$  and confidence level  $1 - \delta$ . More precisely, denote

$$N(\epsilon, \delta) := N_{\mathcal{F}, \mathcal{P}}^L(\epsilon, \delta) := \min\{n \geq 1: \sup_{P \in \mathcal{P}} \mathbb{P}\{\|P_n - P\|_{\mathcal{F}} \geq \epsilon\} \leq \delta\},$$

where  $\|\cdot\|_{\mathcal{F}}$  is the sup-norm in the space  $\ell^\infty(\mathcal{F})$  of all uniformly bounded functions on  $\mathcal{F}$ . Let us call the quantity  $N(\epsilon; \delta)$  the (lower) sample complexity of empirical approximation on the class  $\mathcal{F}$ .

Unfortunately, the quantity  $N_{\mathcal{F}, \mathcal{P}}^L(\epsilon, \delta)$  is not known for most of the non-trivial examples of function classes and only rather conservative upper bounds for this quantity are available. These bounds are expressed in terms of various entropy characteristics and combinatorial quantities, such as VC-dimensions, which themselves are not always known precisely and replaced by the upper bounds [11].

Going back to our control motivation, we note that our problem involves also the finding of the minimum of a certain performance objective or more precisely, finding the controller parameters which will correspond to such minimum. What these randomized algorithms provide is a probably approximate near minimum of a stochastic process  $R$  with confidence  $1 - \delta$ , level  $\alpha$  and accuracy  $\epsilon$  (see [11]).

**Definition 1.** Suppose  $R : \mathcal{Y} \rightarrow \mathbb{R}$ , that  $Q$  is a given probability measure on  $\mathcal{Y}$ , and that  $\alpha \in (0, 1)$ ,  $\delta \in (0, 1)$  and  $\epsilon > 0$  are given. A number  $R_0$  is a probably approximate near minimum of  $R$  with confidence  $1 - \delta$ , level  $\alpha$  and accuracy  $\epsilon$ , if

$$\mathbb{P} \left\{ \inf_{Y \in \mathcal{Y}} R(Y) - \epsilon \leq R_0 \leq \inf_{Y \in \mathcal{Y} \setminus S} R(Y) + \epsilon \right\} \geq 1 - \delta$$

with some measurable set  $S \subseteq \mathcal{Y}$  such that  $Q(S) \leq \alpha$ .

An interpretation of the definition is that we are not searching for the minimum over all of the set  $\mathcal{Y}$  but only over its subset  $\mathcal{Y} \setminus S$ , where  $S$  has a small measure. Unless the actual infimum  $R^*$  is attained in the exceptional set  $S$ ,  $R_0$  is within  $\epsilon$  from the actual infimum with confidence  $1 - \delta$ . Although using Monte Carlo type minimization, it is unlikely to obtain a better estimate of  $R^*$  than  $R_0$ , nothing can be said in practice about the size of the difference  $R_0 - R^*$ . In [11], two algorithms are proposed in order to find this approximate minimum; a first one which only relies on standard Chernoff bounds [3], and a second one which requires the introduction of new learning concepts and combinatorial quantities, i.e. the VC-dimension and the P-dimension.

### 3. Sequential learning algorithms

In this section, we suggest sequential algorithms for a general problem of empirical risk minimization. This approach does not depend on the explicit calculation of the VC-dimension. The sequential algorithms chosen are based on *Rademacher bootstrap* although other bootstrap techniques, developed in statistics (for instance, standard Efron bootstrap or various versions of weighted bootstrap), can be also adopted for our purposes. An important feature of our approach is the randomness of the sample size for which a given accuracy of learning is achieved with a guaranteed probability. Thus, the sample complexity of our method of learning is rather a random variable. Its value is not known in advance and is to be determined in the process of learning. The lower bound for this random variable is the value of the sample size which the sequential learning algorithm starts working with. The upper bounds for the random sample complexity are of the same order of magnitude as the standard conservative upper bounds for the sample complexity of empirical risk minimization algorithms. Thus, *in the worst case* the sequential method of learning would take as much time (up to a numerical constant) as the standard methods do.

The sequential method of learning, described below, is designed to overcome some of the difficulties encountered with the standard learning methods. We start with some basic definitions.

**Definition 2.** Let  $\{\Sigma_n\}_{n \geq 1}$  be a filtration of  $\sigma$ -algebras (i.e., for all  $n \geq 1$ ,  $\Sigma_n \subset \Sigma_{n+1}$ ) such that  $\Sigma_n \subset \Sigma$ ,  $n \geq 1$  and  $X_n$  is  $\Sigma_n$ -measurable. Less formally,

$\Sigma_n$  consists of the events that occur by time  $n$  (in particular, the value of random variable  $X_n$  is known by time  $n$ ).

**Definition 3.** A random variable  $\tau$ , taking positive integer values, will be called a stopping time iff, for all  $n \geq 1$ , we have  $\{\tau = n\} \in \Sigma_n$ . In other words, the decision whether  $\tau \leq n$ , or not, depends only on the information available by time  $n$ .

Given  $\epsilon > 0$  and  $\delta \in (0, 1)$ , let  $\bar{n}(\epsilon, \delta)$  denote the initial sample size of our learning algorithms. We assume that  $\bar{n}$  is a non-increasing function in both  $\epsilon$  and  $\delta$ . Denote by  $\mathcal{T}(\epsilon, \delta) := \mathcal{T}_{\mathcal{F}, \mathcal{P}}(\epsilon, \delta)$  the set of all stopping times  $\tau$  such that  $\tau \geq \bar{n}(\epsilon; \delta)$  and

$$\sup_{P \in \mathcal{P}} \mathbb{P}\{\|P_\tau - P\|_{\mathcal{F}} \geq \epsilon\} \leq \delta.$$

If now  $\tau \in \mathcal{T}(\epsilon, \delta)$  and  $\hat{f} := f_{P_\tau}$  is a function that minimizes the empirical risk based on the sample  $(X_1, \dots, X_\tau)$  it is possible to prove that

$$\sup_{P \in \mathcal{P}} \mathbb{P}\{R_P(f_{P_\tau}) \geq \inf_{f \in \mathcal{F}} R_P(f) + 2\epsilon\} \leq \delta.$$

The questions, though, are how to construct a stopping time from the set  $\mathcal{T}(\epsilon, \delta)$ , based only on the available data (without using the knowledge of  $P$ ) and which of the stopping times from this set is best used in the learning algorithms. The following definition could be useful in this connection:

**Definition 4.** A parametric family of stopping times  $\{v(\epsilon, \delta): \epsilon > 0, \delta \in (0, 1)\}$  is called strongly (statistically) efficient for the class  $\mathcal{F}$  with respect to  $\mathcal{P}$  if and only if (iff) there exist constants  $K_1 \geq 1$ ,  $K_2 \geq 1$  and  $K_3 \geq 1$  such that for all  $\epsilon > 0$  and  $\delta \in (0, 1)$ ,

$$v(\epsilon, \delta) \in \mathcal{T}(K_1\epsilon, \delta)$$

and for all  $\tau \in \mathcal{T}(\epsilon, \delta)$ ,

$$\sup_{P \in \mathcal{P}} \mathbb{P}\{v(K_2\epsilon, \delta) > \tau\} \leq K_3\delta.$$

Thus, using strongly efficient stopping time  $v(\epsilon; \delta)$  allows one to solve the problem of empirical approximation with confidence  $1 - \delta$  and accuracy  $K_1\epsilon$ . With probability at least  $1 - K_3\delta$ , the time required by this algorithm is less than the time needed for *any* sequential algorithm of empirical approximation with accuracy  $\epsilon/(K_1K_2)$  and confidence  $1 - \delta$ .

**Definition 5.** We call a family of stopping times  $\{v(\epsilon, \delta): \epsilon > 0, \delta \in (0, 1)\}$  weakly (statistically) efficient for the class  $\mathcal{F}$  with respect to  $\mathcal{P}$  iff there exist constants  $K_1 \geq 1, K_2 \geq 1$  and  $K_3 \geq 1$  such that for all  $\epsilon > 0$  and  $\delta \in (0, 1)$ ,

$$v(\epsilon, \delta) \in \mathcal{T}(K_1\epsilon, \delta)$$

and

$$\sup_{P \in \mathcal{P}} \mathbb{P}\{v(K_2\epsilon, \delta) > N(\epsilon; \delta)\} \leq K_3\delta.$$

Using weakly efficient stopping time  $v(\epsilon; \delta)$  also allows one to solve the problem of empirical approximation with accuracy  $K_1\epsilon$  and confidence  $1 - \delta$ . With probability at least  $1 - K_3\delta$ , the time required by this algorithm, is less than the sample complexity of empirical approximation with accuracy  $\epsilon/(K_1K_2)$  and confidence  $1 - \delta$ .

Note that, under the assumption  $N(\epsilon; \delta) \geq \bar{n}(\epsilon; \delta)$ , we have  $N(\epsilon, \delta) \in \mathcal{T}(\epsilon, \delta)$ . Hence, any strongly efficient family of stopping times is also weakly efficient.

We show below how to construct efficient stopping times for empirical risk minimization problems. The construction is based on a version of bootstrap. Let  $\{r_n\}_{n \geq 1}$  be a Rademacher sequence (i.e. a sequence of i.i.d. random variables taking values  $+1$  and  $-1$  with probability  $1/2$  each). We assume, in addition, that this sequence is independent of the observations  $\{X_n\}_{n \geq 1}$ . Suppose that (with  $\lfloor \cdot \rfloor$  denoting the floor of the argument)

$$\bar{n}(\epsilon, \delta) \geq \left\lfloor \frac{4}{\epsilon^2} \log \left( \frac{2}{\delta(1 - e^{-\epsilon^2/4})} \right) \right\rfloor + 1.$$

Let

$$v(\epsilon, \delta) := v_{\mathcal{F}}(\epsilon, \delta) := \min \left\{ n \geq \bar{n}(\epsilon, \delta) : \left\| n^{-1} \sum_{j=1}^n r_j \delta_{X_j} \right\|_{\mathcal{F}} \leq \epsilon \right\},$$

where  $\delta_x(f) := f(x)$ . Note that for all  $\epsilon > 0$  and for all  $\delta \in (0, 1)$ ,  $v(\epsilon, \delta)$  is a stopping time and it can be computed by Monte Carlo simulation of the sequence  $\{r_j\}_{j \geq 1}$ .

**Theorem 1** (Koltchinskii et al. [5]).  $\{v_{\mathcal{F}}(\epsilon, \delta) : \epsilon > 0, \delta \in (0, 1)\}$  is a strongly efficient family of stopping times for any class  $\mathcal{F}$  of measurable functions from  $S$  into  $[0, 1]$  with respect to the set  $\mathcal{P}(\mathcal{S})$  of all probability distributions.

The initial time of the previous algorithm could be too large if  $\epsilon$  is very small. Here we construct another version of sequential risk minimization algorithm with smaller initial time.

Define

$$v(\epsilon, \delta) := v_{\mathcal{F}}(\epsilon, \delta) := \min \left\{ n : \left\| n^{-1} \sum_{j=1}^n r_j \delta_{X_j} \right\|_{\mathcal{F}} \leq \epsilon, n := n_k := 2^k \bar{n}(\epsilon, \delta), k = 0, 1, \dots \right\} \quad (2)$$

**Theorem 2** (Koltchinskii et al. [5]). *Suppose that*

$$\bar{n}(\epsilon, \delta) \geq \left\lfloor \frac{4}{\epsilon^2} \log \left( \frac{4}{\delta} \right) \right\rfloor + 1.$$

*Then, for all  $\epsilon > 0$ ,  $\delta \in (0, 1)$ ,  $v(\epsilon; \delta) \in \mathcal{F}(K_1\epsilon; \delta)$  with  $K_1 = 5$ . Moreover, suppose that*

$$N(\epsilon, \delta) \geq \bar{n}(\epsilon, \delta) \geq \left\lfloor \frac{4}{\epsilon^2} \log \left( \frac{4}{\delta} \right) \right\rfloor + 1.$$

*Then  $\{v_{\mathcal{F}}(\epsilon, \delta): \epsilon > 0, \delta \in (0, 1/2)\}$  is a weakly efficient family of stopping times for any class  $\mathcal{F}$  of measurable functions from  $S$  into  $[0, 1]$  with respect to the set  $\mathcal{P}(S)$  of all probability distributions on  $S$ .*

Based on our sequential learning algorithm, a probably approximate near minimum of  $f$  with confidence  $1 - \delta$ , level  $\alpha$  and accuracy  $\epsilon$ , can be found with the following algorithm:

**Algorithm 1.** Given:

- Sets  $\mathcal{X}$  and  $\mathcal{Y}$ ,
- Probability measures  $P$  on  $\mathcal{X}$  and  $Q$  on  $\mathcal{Y}$ ,
- A measurable function  $f: \mathcal{X} \times \mathcal{Y} \rightarrow [0, 1]$ , and
- An accuracy parameter  $\epsilon \in (0, 1)$ , a level parameter  $\alpha \in (0, 1)$ , and a confidence parameter  $\delta \in (0, 1)$ .

Let  $R_p(f(\cdot)) = \mathbb{E}_p[f(X, \cdot)]$  and

$$R_{p_n}(\cdot) = \frac{1}{n} \sum_{j=1}^n f(X_j, \cdot).$$

Then,

1. Choose  $m$  controllers with random parameters having distribution  $Q$  where

$$m \geq \frac{\log(2/\delta)}{\log[1/(1-\alpha)]}.$$

2. Choose  $n$  plants with random parameters having distribution  $P$ , where

$$n = \left\lfloor \frac{4K_1^2}{\epsilon^2} \log \left( \frac{8}{\delta} \right) \right\rfloor + 1.$$

3. Evaluate the stopping variable

$$\gamma = \max_{1 \leq i \leq m} \left| \frac{1}{n} \sum_{j=1}^n r_j f(X_j, Y_i) \right|,$$

where  $r_j$  are Rademacher random variables, i.e. independent identically distributed random variables (also independent of the plant sample) taking

values +1 and -1 with probability 1/2 each. If  $\gamma > \epsilon/K_1$ , add  $n$  more independent plants with parameters having distribution  $P$  to the plant samples, set  $n := 2n$  and repeat step 3.

4. Choose the controller which minimizes the cost function  $R_{P_n}$ . This is the sub-optimal controller in the sense defined above.

#### 4. Applications to control design

In this example we consider the control problem presented by Vidyasagar in [12] and solved via randomized algorithms. This will allow us to illustrate our method and to compare it to the one proposed in [12].

The example concerns the design of an inner-loop controller for the longitudinal axis of an aircraft. The problem is to minimize the weighted sensitivity function over a certain set of uncertain plants, given some constraints on the nominal plant. For further details, the reader is referred to [12].

The closed-loop system is shown in Fig. 1. The plant  $G(s, X)$  is in the form

$$\begin{aligned} \dot{x} &= Ax + Bu, \\ y &= Cx, \end{aligned}$$

where

$$A = \begin{bmatrix} Z_x & 1 - Z_q \\ M_x & M_q \end{bmatrix}, \quad B = \begin{bmatrix} Z_{\delta e} \\ M_{\delta e} \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

The parameters of the matrices have Gaussian distribution with means and standard deviations as in Table 1. In the following, we let  $X = [Z_x \ Z_q \ M_x \ M_q \ Z_{\delta e} \ M_{\delta e}]^T$ .

The transfer function  $HW(s)$  models the different hardware components, such as the sensors, the actuators, the structural filters, etc. It is given by

$$HW(s) = \frac{0.000697s^2 - 0.0397s + 1}{0.000867s^2 + 0.0591s + 1}.$$

We will denote by  $G_0(s)$  the nominal plant and by  $\hat{G}(s)$  (respectively  $\hat{G}_0(s)$ ) the series connection  $G(s)HW(s)$  (respectively  $G_0(s)HW(s)$ ).

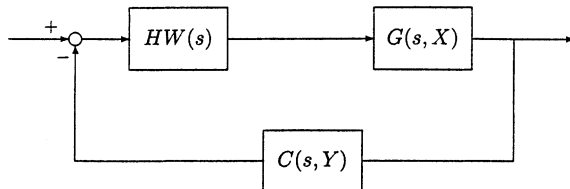


Fig. 1. The closed-loop system.

Table 1  
Parameters for the aircraft model

Parameter	Mean	S.D.
$Z_x$	-0.9381	0.0736
$Z_q$	0.0424	0.0035
$M_z$	1.6630	0.1385
$M_q$	-0.8120	0.0676
$Z_{\delta_e}$	-0.3765	0.0314
$M_{\delta_e}$	-10.8791	3.4695

We choose the controller to have the following structure:

$$C(s) = \left[ -K_a - K_q \frac{(1 + s\tau_1)}{(1 + s\tau_2)} \right],$$

where the four parameters  $K_a, K_q, \tau_1$  and  $\tau_2$  have uniform distributions in the ranges

$$K_a \in [0, 2], \quad K_q \in [0, 1], \quad \tau_1 \in [0.01, 0.1], \quad \tau_2 \in [0.01, 0.1].$$

We thus let  $Y = [K_a \ K_q \ \tau_1 \ \tau_2]^T$ . Our objective is to find the controller which solves the following problem:

$$\min \left\| W(I + \hat{G}C)^{-1} \right\|_{\infty} \quad \text{s.t.} \quad \left\| \frac{0.75C\hat{G}_0}{1 + 1.25C\hat{G}_0} \right\|_{\infty} \leq 1,$$

where the weighting function  $W(s)$  is given by

$$W(s) = \begin{bmatrix} \frac{2.8*6.28*31.4}{(s+6.28)(s+31.4)} & 0 \\ 0 & \frac{2.8*6.28*3.14}{(s+6.28)(s+31.4)} \end{bmatrix}.$$

In order to adopt a randomized algorithm solution, in [12], this problem has been reformulated in the following way. Let us define a cost function

$$\Psi(Y) = \max\{\psi_1(Y), \psi_2(Y)\},$$

where

$$\psi_1(Y) = \begin{cases} 1 & \text{if } \left\| \frac{0.75C\hat{G}_0}{1+1.25C\hat{G}_0} \right\|_{\infty} > 1 \\ 0 & \text{otherwise} \end{cases}$$

and

$$\psi_2(Y) = E_P(\zeta(X, Y))$$

with

$$\zeta(X, Y) = \begin{cases} 1 & \text{if } (\hat{G}(X), C(Y)) \text{ is unstable,} \\ \frac{\|W(I+\hat{G}(X)C(Y))^{-1}\|_{\infty}}{1+\|W(I+\hat{G}(X)C(Y))^{-1}\|_{\infty}} & \text{otherwise.} \end{cases}$$

In our example, the procedure outlined in Algorithm 1 stopped after one iteration, i.e.  $k = 1$ . Therefore, for  $\delta = 0.01$ ,  $\alpha = 0.1$  and  $\epsilon = 0.1$ ,  $m$  evaluated to 51 controllers and  $n$  evaluated to 66,848 plants. In Fig. 2, the stopping condition is shown versus  $n$ . The parameters of the optimal controller are

$$K_a = 1.7826, \quad K_q = 0.7621, \quad \tau_1 = 0.0511, \quad \tau_2 = 0.0117,$$

and the corresponding value of the cost function is  $\Psi(Y_{\text{opt}}) = 0.7149$ , which compares favorably with the results of [12], where 2,619,047 plants were needed to achieve  $\Psi(Y_{\text{opt}}) = 0.7684$  with the same  $\epsilon, \alpha, \delta$  and  $m$ .

**Remark 1.** As shown by Fig. 2, the stopping condition is met far before the minimum number of plant samples  $n$ . This hints that with the same number of samples  $n$ , problems with much higher  $P$ -dimension could be addressed. The  $P$ -dimension of this problem was evaluated in [12] and is equal to  $d = 118$ . Therefore other types of controller could be used instead of the first-order one we used. For instance the order of the controller could be increased until certain performance, in term of the desired value for  $\Psi_{\text{opt}}$ , is achieved [5].

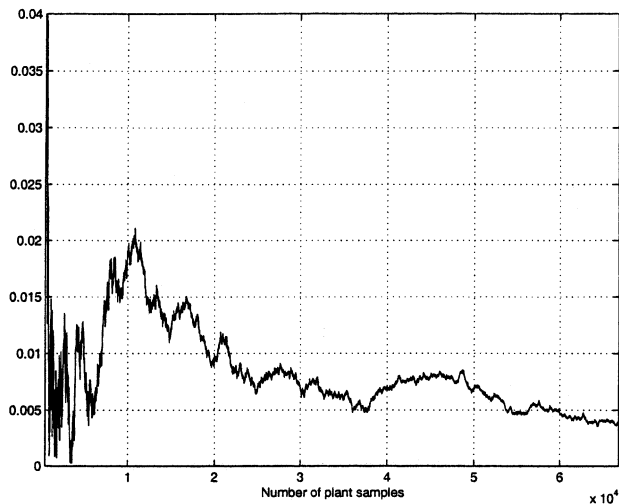


Fig. 2. The stopping variable.

## 5. Conclusions

In this paper we have drastically reduced the number of samples needed in order to obtain performance guarantees in robust control synthesis problems. This reduction is achieved by introducing sequential bootstrapping algorithms and exploiting the fact that the sample complexity is itself a random variable. This has allowed us to present Algorithm 1 as an efficient design methodology for fixed-order robust control design problems [5].

It should be noted that the methodology presented in this paper can be used in many other application areas: one only needs to have an efficient analysis tool in order to convert it to an efficient design methodology. This is due to the fact that the design problem is converted to a sequence of analysis or verification problems after sampling more plants and controllers than the minimum number required by Algorithm 1. It should also be noted that the computational complexity or the undecidability of the problems studied are not eliminated but only avoided by relaxing the design question.

The randomized algorithms approach may be applied to design fixed-structure controllers for non-linear systems (see for example the Pfaffian systems discussed in [10]), for discrete-time systems, and to building software systems for practical control design problems. It may also be used for problems of hybrid and intelligent control where events and dynamics are mixed. Our future research is concentrating at the theoretical level in obtaining better optimization algorithms and at the application level in designing software modules for linear and non-linear control design. We are in the process of extending the methodology using concepts from support vector machines [8] to increasing-complexity controllers. We are also investigating the applicability of the statistical learning approach in combination with the “unfalsified controller” design discussed in [7].

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