J. Symbolic Computation (1995) **11**, 1–6

Application of Quantifier Elimination Theory to Robust Multi-objective Feedback Design

P. Dorato†, Wei Yang†, and C. Abdallah†

†*Department of Electrical and Computer Engineering, University of New Mexico*

Albuquerque, NM 87131-1356, USA

(Received 30 September 1995)

This paper shows how certain robust multi-objective feedback design problems can be reduced to quantifier elimination (QE) problems. In particular it is shown how robust stabilization and robust frequency domain performance specifications can be reduced to systems of polynomial inequalities with suitable logic quantifiers, $\forall$ and $\exists$. Because of computational complexity the size of problems that can solved by QE methods is limited. However the design problems considered here do not have *analytical* solutions, so that even the solution of modest sized problems may be of practical interest.

1. Introduction

In 1975 Anderson *et al.* (Anderson & Bose & Jury 1975) proposed the application of Tarski-Seidenberg decision theory (Tarski 1951, Seidenberg 1954) for the solution of the *static output feedback stabilization problem.* The static output feedback stabilization problem is one of the most important open problems in feedback design. The problem can be stated mathematically as follows: find a matrix K such that all of the eigenvalues of the matrix $A + BKC$ have negative real- parts, given the matrices A, B and C. This problem has no general analytical solution. By use of the Liénard-Chipart criterion (Gantmacher 1959), the problem can be reduced to a system of polynomial inequalities in the coefficients of the matrix K. The computational complexity and lack of software severely limited the interest in the results presented by Anderson *et al.* in 1975. However since then some improved algorithms have been developed (Collins 1975), (Collins & Hong 1991), and implemented (Quantifier Elimination by Partial Cylindrical Algebraic Decomposition-QEPCAD, Hong 1992). In light of of new developments in quantifier elimination theory, we explore here the application of the theory to a class of feedback design problems that is of great practical interest, that is robust multi-objective design.

2. The Robust Multi-objective Feedback Design Problem

Figure 1 shows a typical feedback control configuration. We assume here single-input-single-out linear time-invariant systems. The *plant* , i.e. object being controlled, is characterized by its Laplace transfer function $G(s, p)$, where s is the Laplace transform variable

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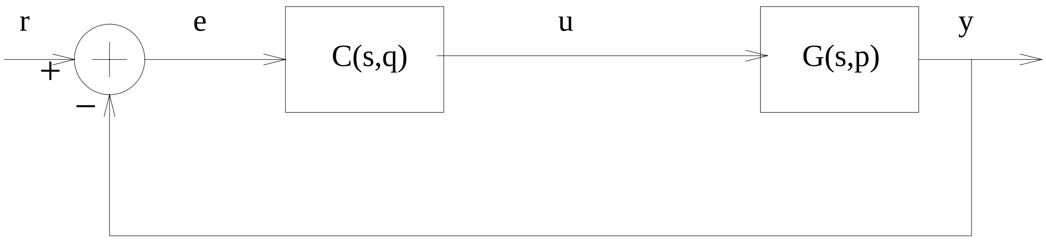

Figure 1. Block Diagram Of Feedback System

and p is a vector of plant parameter values. The *compensator* $C(s,q)$, the feedback transfer function to be designed, is assumed to be of fixed structure and to include of vector of design parameters q. Both transfer functions are assumed to be rational functions in the variable s, and the components of the vectors p and q are assumed to enter in the coefficients of the s-polynomials as polynomial functions.

Most realistic feedback design problems are characterized by uncertainty in plant parameter values and multiple design objectives. The term *robust* is used to indicate that the design objectives are met *for all admissible plant parameter values.* We list below some typical design objectives.

Closed-loop Stability. In order to guarantee the stability of the closed-loop system all the zeros of the rational function

$$1 + C(s,q)G(s,p)$$

must have negative real parts. If this rational function is expressed as the ratio of two polynomials, i.e. $N(s)/D(s)$, then all the zeros of the numerator polynomial $N(s)$ must have negative real-parts. A polynomial with this property is commonly referred to as a *Hurwitz* polynomial.

Tracking Error. A major feedback design objective is the minimization of the tracking error $e(t)$ (See Figure 1). The transfer function relating the command input $r(t)$ to the error $e(t)$ is given by

$$S(s) = E(s)/R(s) = \frac{1}{1 + C(s,q)G(s,p)}$$

Acceptable levels of tracking error may be specified in the frequency domain by the inequality condition

$$\mid S(j\omega) \mid < \alpha_T,\ 0 \le \omega \le \omega_1$$

Control Effort. Another important feedback design objective is the maintenance of the control input $u(t)$ within acceptable levels. The transfer function relating the command input $r(t)$ to the control signal $u(t)$ is given by

$$W(s) = U(s)/R(s) = \frac{C(s,q)}{1 + C(s,q)G(s,p)}$$

Acceptable levels of control effort may be specified in the frequency domain by the inequality condition

$$\mid W(j\omega) \mid < \alpha_U,\ \text{for all } \omega$$

We define here the following *robust multi-objective feedback design problem*: find design vectors q such that a set of performance objectives such as listed above are met for all plant parameter values that are inside "uncertainty" intervals

$$\underline{p}_i \le p_i \le \overline{p}_i \tag{2.1}$$

where p_i denotes the i^{th} component of the vector p. The set of plant vectors defined in (2.1) will be denoted $\mathcal{P}$.

3. Quantifier Elimination Solution

We will demonstrate in this section that all of the performance objectives listed in the previous section can be expressed as quantified polynomial inequalities. With the additional assumption that any real number in these inequalities can be approximated by a rational number, the problem is then reduced to a quantifier elimination problem that can be "decided" with a finite number of algebraic operations.

Consider the closed-loop stability requirement. If the polynomial $N(s)$ is written

$$N(s) = a_0 s^n + a_1 s^{n-1} + a_2 s^{n-2} + ... + a_n$$

then the Liénard-Chipart conditions for $N(s)$ to be a Hurwitz polynomial are (Gantmacher 1959) given by the inequalities

$$a_n > 0, a_{n-2} > 0, ...; \Delta_1 > 0, \Delta_3 > 0, ...$$

or

$$a_n > 0, a_{n-2} > 0, ... : \Delta_2 > 0, \Delta_4 > 0, ...$$

where Δ_i is the so-called Hurwitz determinant of order i given by

$$\Delta_i \quad = \quad \det \begin{bmatrix} a_1 & a_3 & a_5 & \cdots \\ a_0 & a_2 & a_4 & \cdots \\ 0 & a_1 & a_3 & \cdots \\ 0 & a_0 & a_2 & \cdots \\ \vdots & \vdots & \vdots & \vdots \\ \cdots & \cdots & \cdots & a_i \end{bmatrix}, \; a_k = 0 \text{ for } k > n$$

If the parameters q_i and p_i enter the coefficients of $N(s)$ as polynomial functions, then the above inequalities will be polynomial inequalities. Let the resulting polynomial inequalities be denoted, $u_i(p, q) > 0$, then robust closed-loop stability requires the truth of the quantified formula

$$\forall (p)[p \in \mathcal{P} \rightarrow u_i(p, q) > 0] \tag{3.1}$$

The tracking and control objectives may be reduced to quantified polynomial inequalities by noting that conditions of the form

$$\mid A(j\omega)/B(j\omega) \mid < \alpha$$

where $A(s)$ and $B(s)$ are polynomials in s may be written

$$\mid A(j\omega) \mid^2 < \alpha^2 \mid B(j\omega) \mid^2 \tag{3.2}$$

Clearly inequality (3.2) may be written in the form, $v(p,q,\omega) \geq 0$, where $v(p,q,w)$ is polynomial in the variables p_i, q_i and ω. Thus the robust performance requirements require the truth of the quantified formula

$$\forall(p)\forall(\omega)[(p \in \mathcal{P} \wedge \omega \in \Omega) \rightarrow v_1(p,q,\omega) > 0 \wedge v_2(p,q,\omega) > 0 \wedge ...] \tag{3.3}$$

where Ω represents the frequency interval of interest, and the polynomial functions $v_i(p,q,\omega)$ result from performance objectives such as tracking and control effort. Note that if the frequency intervals of interest differ for the various performance objectives, more than one frequency variable will have to be introduced in the formula (3.3). If QE theory is used to eliminate the quantifiers in (3.1) and (3.3), one obtains a quantifier-free formula in the design vector q, i.e. $\Psi(q)$. This formula defines the set of design vectors q which robustly meet all the design objectives. The question of existence of a solution may be settled by applying the $\exists$ quantifier on the the formula $\Psi(q)$ and using QE to eliminate this quantifier.

4. Design Example

The problem considered here is a simplified version of the problem in (Fiorio *et al.* 1993). The plant is assumed to be an unstable first-order system with transfer function

$$G(s,p) = \frac{p_1}{1 - s/p_2}, \quad 0.8 \leq p_{1,2} \leq 1.25 \tag{4.1}$$

with simple output feedback $C(s,q) = q_1$. The tracking error bound (See section 2) is assumed to be given by $\alpha_T = 0.1$, with $\omega_1 = 10$, and the control effort bound is given to be $\alpha_U = 20$. The admissible set of plant parameters is re-written $16 \leq 20p_{1,2} \leq 25$ to meet the integer-coefficient requirement of the QE theory. To solve this robust multi-objective problem the following polynomial inequalities must be satisfied for all admissible plant parameters.

Robust stability: The stability criterion for the first-order closed-loop characteristic polynomial is simply,

$$u_1(p,q) = -p_2(1 + p_1 q_1) > 0 \tag{4.2}$$

Tracking error: If the magnitude-squared of $S(j\omega)$ is computed, and the denominator polynomial is cleared we obtain the condition,

$$v_1(p,q,\omega) = 99\omega^2 + (p_2)^2(100(1 + p_1 q_1)^2 - 1) > 0, \quad 0 \leq \omega \leq 10 \tag{4.3}$$

Control effort: With the same computations as for the tracking error, we obtain for control effort the condition,

$$v_2(p,q,\omega) = (400 - q_1^2)\omega^2 + (p_2)^2(400(1 + p_1 q_1)^2 - q_1^2) > 0, \text{all real } \omega \tag{4.4}$$

QEPCAD software produced the following quantifier-free formula

$$\Psi(q_1) = [(q_1 + 20 \geq 0) \wedge (q_1 + 2 \leq 0] \vee [(8q_1 + 11 < 0) \wedge (q_1 + 2 \geq 0)] \tag{4.5}$$

¿From (4.5) one obtains the following parameterization of of all compensators which satisfy the robust multi-objective problem posed above.

$$C(s,q_1) = q_1, \quad -20 \leq q_1 < -1.375$$

```
[33 CDC -Milanese et.al]
(q1,p1,p2,w1,w2)
1
(A p1)(A p2)(A w1)(A w2)
[
        [16 <= 20 p1 /\ 20 p1 <= 25 /\
         16 <= 20 p2 /\ 20 p2 <= 25 /\ 0 <= w1 /\ w1 <= 10 ]
        ==>
        [p2 (1 + p1 q1) < 0 /\
99 w1^2 + p2^2 (100 (1 + p1 q1)^2 - 1 ) > 0 /\
(400 - q1^2) w2^2 + p2^2 (400 (1 + p1 q1)^2 - q1^2) > 0
        ]
].
go
go
go
go
```

Table 1. Input File for QEPCAD

To illustrate QEPCAD syntax for this particular example, we list the QEPCAD input file in Table 1. Note that the first step in defining the inputs is to list all the variables, e.g. $(q_1, p_1, p_2, w1, w2)$, with the un-quantified variables listed first. The un-quantified variables are identified by listing the number of such variables in the variable list. In this particular example the number "1" is listed since there is only one un-quantified variable, q_1. Note also that the variable ω is specified as two separate variable $w1$ and $w2$, since the inequalities involving ω are for different ranges of ω.

In (Fiorio *et al.* 1993) a more complicated compensator was considered for this problem, i.e. a proportional-plus-integral (PI) compensator of the form

$$C(s, q) = q_1 \frac{1 + s/q_2}{s}$$

In (Fiorio *et al.* 1993), Bernstein polynomials are used to obtain an "approximate" region for the design parameters q_1 and q_2. Existing QE software was not able to solve this more complicated problem. However it should be noted that QE theory attempts to find an "exact" solution to the problem.

5. Conclusions

We have shown how some difficult robust multi-objective feedback design problems can be reduced to quantifier-elimination problems. The design example presented here illustrates the fact that solutions can be obtained with existing QE algorithms and software to at least some practical problems. However the example also illustrates that it does not take much complexity to saturate existing QE software. As discussed in (Basu *et al.* 1994), it is known that in certain variables QE algorithms are doubly exponential in complexity. It is also know (Nemirovskii 1993) that many problems of *robust stability analysis*, where the compensator parameters are pre-specified, are $\mathcal{NP}$ hard. Still for some problems QE methods may provide solution that would be difficult to obtain by brute-force deterministic or stochastic (Monte Carlo) discretization methods.

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